* Product of Three vectors :-

(A) Scalar Triple Product :-

$$\vec{a}\cdot[\vec{b}\times\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\vec{a}\cdot[\vec{b}\times\vec{c}] = \vec{b}\cdot[\vec{c}\times\vec{a}] = \vec{c}\cdot[\vec{a}\times\vec{b}]$$

Thus,

$$[\vec{A}\vec{B}\vec{C}] = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$= (A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= A_x[B_yC_z - C_yB_z] + A_y[C_xB_z - C_zB_x] + A_z[B_xC_y - B_yC_x]$$

or

$$[\vec{A}\vec{B}\vec{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = [\vec{C}\vec{A}\vec{B}] =$$

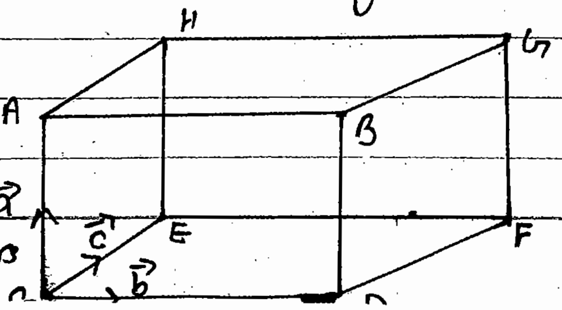
$$\begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix} = [\vec{B}\vec{C}\vec{A}]$$

Thus we have $[\vec{A}\vec{B}\vec{C}] = [\vec{C}\vec{A}\vec{B}] = [\vec{B}\vec{C}\vec{A}]$

or $\vec{A}\cdot[\vec{B}\times\vec{C}] = \vec{C}\cdot[\vec{A}\times\vec{B}] = \vec{B}\cdot[\vec{C}\times\vec{A}]$

Thus dot and cross may be inter-changed at will.

$\vec{A}\cdot(\vec{B}\times\vec{C})$ or the scaler triple product of three vectors is equal to the Volume of parallelopiped having vectors as concurrent edges



Let three vectors $\vec{a}, \vec{b}, \vec{c}$ are taken on a parallelopiped having vectors $\vec{a}, b, \vec{c}$ as concurrent edges.

Then,

$$[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}.[\vec{b} \times \vec{c}]$$

So $$V = \Big|\underbrace{\vec{a}}_{\text{hight}}.[\underbrace{\vec{b} \times \vec{c}}_{\text{base Area}}]\Big|$$ {Volume = hight x Base Area.}

When the three vectors lie in a plane the volume of the parallelopiped is zero. hence, for three vectors to be coplanar, their scalar product vanishes.

[B] Vector Triple Product :-

$\vec{A} \times (\vec{B} \times \vec{C})$ is called vector triple product.

Since the vector $(\vec{B} \times \vec{C})$ is normal to the plane containing $\vec{B}$ and $\vec{C}$. Likewise, the vector $\vec{A} \times (\vec{B} \times \vec{C})$ is normal to the plane containing $\vec{A}$ and $(\vec{B} \times \vec{C})$, i.e. in the same plane as $\vec{B}$ and $\vec{C}$.

let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times [(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})] \\
&= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times [(b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} \\
&\qquad + (b_1c_2 - b_2c_1)\hat{k}] \\
&= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\hat{i} + [a_3(b_2c_3 - b_3c_2) \\
&\qquad - a_1(b_1c_2 - b_2c_1)]\hat{j} + [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\hat{k} \\
&= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3) \\
&\qquad (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \\
&= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}. \text{ Ans}
\end{aligned}$$

A-1

2.15 The equation of the plane that contains the points $P(2,-1,1)$, $Q(3,2,-1)$, $R(-1,3,2)$ is equal to?

(a) $11x+5y+13z=30$ (b) $11x+7y+9z=30$

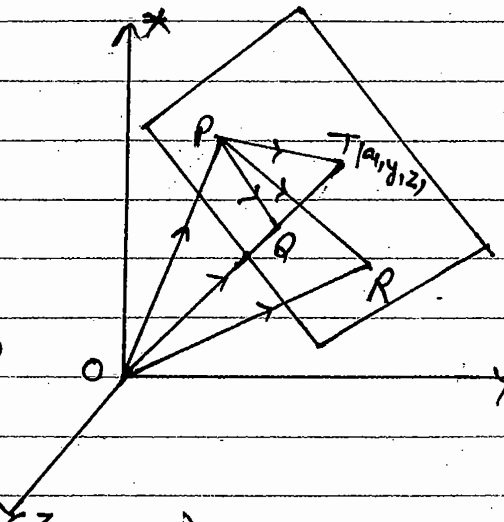
(c) $4x+7y-9z=30$ (d) $4x+11y-3z=30$

Sol

$\vec{P} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{Q} = 3\hat{i}+2\hat{j}-\hat{k}$

$\vec{R} = -\hat{i}+3\hat{j}+2\hat{k}$



$$[\vec{P},\vec{Q},\vec{R}] = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ -1 & 3 & 2 \end{bmatrix} \neq 0$$

$\vec{OP}\cdot(\vec{OQ}\times\vec{OR}) \neq 0$ (not always $=0$)

So

$$\vec{PT}\cdot(\vec{PQ}\times\vec{PR}) = 0$$

$$\vec{PT} = \vec{OT}-\vec{OP} = x\hat{i}+y\hat{j}+z\hat{k} - 2\hat{i}+\hat{j}-\hat{k}$$
$$= (x-2)\hat{i} + (1+y)\hat{j} + (z-1)\hat{k}$$

$$\vec{PQ} = \vec{OQ}-\vec{OP} = (3-2)\hat{i} + (2+1)\hat{j} + (-1-1)\hat{k}$$
$$= \hat{i}+3\hat{j}-2\hat{k}$$

$$\vec{PR} = \vec{OR}-\vec{OP} = (-1-2)\hat{i} + (3+1)\hat{j} + (2-1)\hat{k}$$
$$= -3\hat{i}+4\hat{j}+\hat{k}.$$

$P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$, $R(x_3,y_3,z_3)$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

$$= \hat{i}(3+8) - \hat{j}(1-6) + \hat{k}(4+9)$$

$$= 11\hat{i} + 5\hat{j} + 13\hat{k}$$

$$\{(x-2)\hat{i} + (1+y)\hat{j} + (z-1)\hat{k}\} \cdot 11\hat{i} + 5\hat{j} + 13\hat{k} = 0$$

$$\Rightarrow 11(x-2) + 5(1+y) + (z-1)13 = 0$$

$$\Rightarrow 11x - 22 + 5 + 5y + 13z - 13 = 0$$

$$\Rightarrow 11x + 5y + 13z = 22 + 13 - 5$$

$$\Rightarrow 11x + 5y + 13z = 35 - 5$$

$$\Rightarrow \boxed{11x + 5y + 13z = 30}$$

Ans

Q.17 If $\vec{b} = \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$, then $\vec{b}$ can be simplified to?

(a) 0 (b) $\vec{a}$ (c) $2\vec{a}$ (d) None of these.

Sol^n

$$b = \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j} + \hat{k} \times (\vec{a} \times \hat{k})$$

$$= \vec{a}(\hat{i}.\hat{i}) - \hat{i}(\hat{i}.\vec{a}) + \vec{a}(\hat{j}.\hat{j}) - \hat{j}(\hat{j}.\vec{a}) + \vec{a}(\hat{k}.\hat{k}) - \hat{k}(\hat{k}.\vec{a})$$

$$= 3\vec{a} - \vec{a}$$

$$= 2\vec{a}$$ Ans.

Q.19 Three unit vectors $\vec{a}, \vec{b}, \vec{c}$ ($\vec{b}$ and $\vec{c}$ are not parallel) are such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are?

(a) $30°, 90°$ (b) $150°, 90°$ (c) $60°, 90°$ (d) $90°, 30°$

Sol^n

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c}$$